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# Glauber dynamics on the cycles: Spectral distribution of the generator

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We consider Glauber dynamics on finite cycles. By introducing a vacuum state we consider an algebraic probability space for the generator of the dynamics. We obtain a quantum decomposition of the generator and construct an interacting Fock space. As a result we obtain a distribution of the generator in the vacuum state. We also discuss the monotonicity of the moments of spectral measure as the couplings increase. In particular, when the couplings are assumed to be uniform, as the cycle grows to an infinite chain, we show that the distribution (under suitable dilation and translation) converges to a Kesten distribution.

*Keywords*: Glauber dynamics on cycles; quantum probability; interacting Fock space; quantum decomposition; vacuum state; spectral distribution; Kesten distribution.

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## 1. Introduction

In this paper we study the Glauber dynamics for the ferro-magnetic Ising model on the cycles. Here we emphasize that in our model the nearest neighbor interaction couplings need not be uniform. As a matter of fact, the study for this kind of model has been thoroughly established by Nacu[7] (see also Ref. 8). Our aim is to look at the dynamics from the quantum probabilistic viewpoint.

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Glauber dynamics is well-known to be useful in Markov chain Monte Carlo algorithms and it is a model for the actual dynamics to the equilibrium of the system.[6] The spectral analysis of the model is one of the main studies because it gives an information for the mixing time of the dynamics. The Glauber dynamics draws interests not only for the spin systems on the Euclidean lattices but also for the model on graphs, especially on trees. There are many literature investigating the Glauber dynamics and its properties including mixing condition, spectral gap, logarithmic Sobolev inequality. Here we list just a few number of references and refer the reader to them and references therein. See for example, Refs. 2, 4, 5 and 9.

In this paper we study the Glauber dynamics on cycles. The spin system we consider is a ferro-magnetic Ising model without external fields, but possibly with varying interacting couplings. Nacu has shown that the relaxation time is an increasing function of any of the couplings.[7] For this study, Yuval Peres conjectured that for the ferro-magnetic Ising model on any graph, the relaxation time is an increasing function of the couplings.[7] So, the result of Nacu answers to this question positively on cycles.

Our approach is to use a method in quantum probability theory for the spectral analysis of the generator. It turns out that it is possible to "quantum decompose" the generator of the dynamics and thereby we can use the theory of interacting Fock space. We will represent the moments (w.r.t. the spectral measure) by a vacuum expectation and get a formula of a type of Accardi–Bożejko[3] (Theorem 2.2). As a by-product, we can show that the moments are increasing function of the couplings if the nearby couplings do not vary much relative to each other (Theorem 2.4). Finally, assuming that the couplings are uniform, we consider the growing cycles, and show that the limit distribution of the spectrum, when it is suitably renormalized, is a Kesten distribution (Theorem 3.1).

This paper is organized as follows. In Sec. 2, we introduce the model and give the main results. In Sec. 3, we consider the growing cycles and compute the limit distribution.

## 2. Model and the Main Results

### 2.1. *Glauber dynamics on finite graphs*

Let $G = (V, E)$ be a finite connected graph. On the space of spin configurations $S = \{-1, 1\}^V$, we define a probability measure $\pi$ (Gibbs measure) by

$$\pi(\sigma) := Z^{-1} \exp\left(\sum_{xy\in E} J_{xy}\sigma_x\sigma_y\right), \quad \sigma = (\sigma_x)_{x\in V}. \tag{2.1}$$

Here $Z$ is the normalization constant and the nonnegative constants $J_{xy}$ denote the strength of the interaction. Thus our model is a ferro-magnetic Ising model without external fields and with no boundary conditions. But, we allow the possibility that the interacting couplings may differ along the positions.

The transition matrix $A$ of the (discrete) Glauber dynamics is given by

$$A(\sigma,\eta) = \begin{cases} \dfrac{1}{|V|}\dfrac{\pi(\sigma^x)}{\pi(\sigma)+\pi(\sigma^x)}, & \eta=\sigma^x, \quad x\in V \\ 1-\displaystyle\sum_{x\in V} A(\sigma,\sigma^x), & \eta=\sigma \\ 0, & \text{otherwise,} \end{cases} \tag{2.2}$$

where $\sigma^x \in S$ is the spin configuration obtained from $\sigma$ by flipping a spin at site $x$:

$$(\sigma^x)_y = (1-2\delta_{x,y})\sigma_y.$$

Since $A$ is reversible, that is, it satisfies

$$A(\sigma,\eta)\pi(\sigma) = A(\eta,\sigma)\pi(\eta) \tag{2.3}$$

for all $\sigma,\eta \in S$, it is obvious that $A$ is real in the sense that $A^* = A$ in the Hilbert space $l^2(S,\pi)$, where the inner product is defined by

$$\langle f,g\rangle := \int \overline{f(\sigma)}g(\sigma)\pi(d\sigma), \quad f,g \in l^2(S,\pi). \tag{2.4}$$

It is easy to see that $0 \le A \le I$.

Now let us focus on the cycles. Let the graph $G$ be the cycle $C_n$ of length $n$. It is convenient to label the vertex set $V$ by $V=\{0,1,\dots,n-1\}$ and we will look at it as $\mathbb{Z}_n$ with modulo $n$ (i.e. $n\equiv 0$). The edge set is $E=\{i(i+1): i=0,1,\dots,n-1\}$. For each $i\in V$, define a function $e_i$ on $S$ by

$$e_i(\sigma) := \sigma_i, \quad \sigma \in S. \tag{2.5}$$

It is then easy to see that (see Refs. 7 and 8)

$$Ae_i = \left(1-\frac{1}{n}\right)e_i + \frac{1}{n}(\alpha_i e_{i-1} + \beta_{i+1}e_{i+1}), \tag{2.6}$$

where

$$\alpha_i := \frac{s_{i-1}}{c_{i-1}+c_i} \quad \text{and} \quad \beta_{i+1} := \frac{s_i}{c_{i-1}+c_i}, \tag{2.7}$$

with $s_i := \sinh 2J_{ii+1}$ and $c_i := \cosh 2J_{ii+1}$. In particular, Eq. (2.6) says that the operator $A$ is invariant on the $n$-dimensional subspace $\mathcal{H}_n$ spanned by $\{e_i : i\in V\}$.

In the next subsection, we will discuss the spectral distribution of $A$. For that purpose we will use the one-mode interacting Fock spaces

### 2.2. *Interacting Fock space for the Glauber dynamics*

We would like to investigate the spectrum of $A$. It is convenient to consider the operator

$$B := nA - (n-1)I. \tag{2.8}$$

The spectrum of $B$ is just a dilation of the spectrum of $A$ followed by a translation. By (2.6) $B$ has the representation:

$$Be_i = \alpha_i e_{i-1} + \beta_{i+1} e_{i+1} \tag{2.9}$$

with the coefficients $\alpha_i$ and $\beta_{i+1}$ in (2.7).

From now on we understand the vectors $\{e_0, e_1, \dots, e_{n-1}\}$ as a formal symbol for an orthonormal basis of $n$-dimensional Hilbert space:

$$(e_i, e_j) := \delta_{i,j}, \quad i, j \in \{0, 1, \dots, n-1\}.$$

In the basis $\{e_0, e_1, \dots, e_{n-1}\}$, $B$ has a matrix representation

$$B = \begin{pmatrix} 0 & \alpha_1 & 0 & \cdots & 0 & \beta_0 \\ \beta_1 & 0 & \alpha_2 & \cdots & & \\ 0 & \beta_2 & 0 & \cdots & & \\ & & & \cdots & & \\ 0 & & \cdots & & 0 & \alpha_{n-1} \\ \alpha_0 & 0 & \cdots & & \beta_{n-1} & 0 \end{pmatrix}. \tag{2.10}$$

We notice that $B$ is not symmetric in this representation though it has real eigenvalues. We want to make a symmetrization.

**Proposition 2.1.** *By a similarity transformation, the matrix $B$ can be symmetrized.*

**Proof.** Define

$$a_i := \sqrt{\frac{\prod_{j=1}^{i} \beta_j}{\prod_{j=1}^{i} \alpha_j}}, \quad i = 1, \dots, n-1 \quad \text{and} \quad a_0 := \sqrt{\frac{\prod_{j=0}^{n-1} \beta_j}{\prod_{j=0}^{n-1} \alpha_j}} = 1.$$

Then by defining

$$S := \begin{pmatrix} a_0 & & & \\ & a_1 & & \mathbf{0} \\ & & \ddots & \\ & \mathbf{0} & & \\ & & & a_{n-1} \end{pmatrix},$$

we get

$$\widetilde{B} := S^{-1}BS = \begin{pmatrix} 0 & \sqrt{\omega_1} & 0 & \cdots & 0 & \sqrt{\omega_0} \\ \sqrt{\omega_1} & 0 & \sqrt{\omega_2} & \cdots & & \\ 0 & \sqrt{\omega_2} & 0 & \cdots & & \\ & & & \cdots & & \\ 0 & & \cdots & & 0 & \sqrt{\omega_{n-1}} \\ \sqrt{\omega_0} & 0 & \cdots & & \sqrt{\omega_{n-1}} & 0 \end{pmatrix},$$

where

$$\omega_i = \alpha_i \beta_i, \quad i = 0, 1, \ldots, n-1. \tag{2.11}$$

□

Noticing that the spectrum of $B$ and $\widetilde{B}$ are the same, we next discuss the moments of the spectral measure $\mu_0$ of $\widetilde{B}$ associated with the vector $e_0$. It is known that there is a probability measure such that for all $m = 0, 1, 2, \ldots$, the following equality holds Theorems 1.52 and 1.54 of Ref. 3:

$$M_m(\mu_0) := \int_{-\infty}^{\infty} x^m \mu_0(dx) = (e_0, (\widetilde{B})^m e_0). \tag{2.12}$$

We adopt some techniques used in the quantum probability theory, mainly from the theory of (one-mode) interacting Fock spaces. We first decompose the operator $\widetilde{B}$ as

$$\widetilde{B} = B^+ + B^-, \tag{2.13}$$

where

$$B^+(e_i) = \sqrt{\omega_{i+1}} e_{i+1}, \quad B^-(e_i) = \sqrt{\omega_i} e_{i-1}, \quad i = 0, 1, \ldots, n-1.$$

We call $B^+$ and $B^-$ the creation and annihilation operators, or raising and lowering operators, respectively. In order to compute the moments, since $B^-$ does not "kill" the virtual vacuum vector $e_0$, we need to modify the so-called Accardi–Bożejko formula.[1,3] Notice that the raising and lowering operators can be applied indefinitely without limit. So, it is convenient to consider the system of vectors $\{e_i : i \in \mathbb{Z}\}$ with the convention that $e_i = e_j$ whenever $i = j (\mathrm{mod}\ n)$. When we expand the power $(\widetilde{B})^m = (B^+ + B^-)^m$, a typical term is of the type

$$B^{\varepsilon_m} \cdots B^{\varepsilon_1} \tag{2.14}$$

with $\varepsilon_i \in \{+, -\}$, $i = 1, \ldots, m$. We let $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ and to each operator of the type in (2.14), we associate a sequence of points in $\mathbb{Z}_+ \times \mathbb{Z}$ given by $\{(0,0), (1, s_1), \ldots, (m, s_m)\}$, where $s_i := \sum_{j=1}^{i} \varepsilon_j$ with a convention $+ \equiv +1$ and $- \equiv -1$. By connecting the nearby points in this sequence by a line segment, we may regard it as a connected polygonal path, i.e. a random walk path, say $\gamma$, in the plane. Notice that

$$(e_0, B^{\varepsilon_m} \cdots B^{\varepsilon_1} e_0) \neq 0 \quad \text{if and only if } s_m = 0 \ (\mathrm{mod}\ n),$$

where $s_m$ is the space position at time $m$ in the corresponding random walk path $\gamma$. This is nothing but to saying that $\gamma$, understood as a random walk path on the circle, starts from 0 and ends at 0 at time $m$. The collection of all such paths we denote by $\mathcal{P}_{0,0}(m)$:

$$\mathcal{P}_{0,0}(m) := \{\gamma = \{(0,0), (1, s_1), \ldots, (m, s_m)\} : s_i - s_{i-1} = \pm 1, s_m = 0 \ (\mathrm{mod}\ n)\}.$$

Now suppose that $\gamma = \{(0,0),(1,s_1),\ldots,(m,s_m)\} \in \mathcal{P}_{0,0}(m)$. If $s_m \neq 0$, say $s_m > 0$, by adding a straight line connecting two points $(m,s_m)$ and $(m+s_m,0)$ to the end point, we extend the path to $\widetilde{\gamma} := \{(0,0),(1,s_1),\ldots,(m,s_m),(m+1, s_m-1),\ldots,(m+s_m,0)\}$, which ends at the origin. This gives rise to a non-crossing pair partition $\vartheta \equiv \vartheta(\gamma)$ of $\{1,2,\ldots,m+s_m\}$ in a unique way such that if $v = \{l < r\} \in \vartheta$, then $\varepsilon_l\varepsilon_r = -1$ and $\sum_{k=l}^{r} \varepsilon_k = 0$ (see Refs. 1 and 3). As in Refs. 1 and 3, we define a depth $d_\theta(v)$ for each $v \in \vartheta$, but the depth may have non-positive value because the random walk path can visit the negative positions. The exact definition of depth is given by the following way. Let $v = \{l < r\} \in \vartheta$. Then

$$d_\vartheta(v) = \begin{cases} k+1, & \text{if } \varepsilon_l = + \text{ and there are } k \text{ number of pairs } v' = \{l' < r'\} \in \vartheta \\ & \quad \text{such that } l' < l < r < r' \\ -k, & \text{if } \varepsilon_l = - \text{ and there are } k \text{ number of pairs } v' = \{l' < r'\} \in \vartheta \\ & \quad \text{such that } l' < l < r < r'. \end{cases} \tag{2.15}$$

Now we can give the moments $M_m(\mu_0)$ with the matrix components. The following theorem is a straightforward modification of the Accardi–Bożejko formula.[1,3] We say that a pair $v \in \vartheta(\gamma)$ is *virtual* if $v = \{l < r\}$ is a pair with a point in an enlarged set, i.e. $r > m$ for $\gamma = \{(0,0),(1,s_1),\ldots,(m,s_m)\}$. The other pairs we call *proper*. Below we also use $\omega(i)$ for $\omega_i$ with a slight abuse of notation.

**Theorem 2.2.** *The moments* $M_m(\mu_0)$ *in* (2.12) *can be computed by the following formula.*

$$M_m(\mu_0) = \sum_{\gamma\in\mathcal{P}_{0,0}(m)} \prod_{v\in\vartheta(\gamma)} \omega^*(d_\vartheta(v)), \quad m = 0,1,2,\ldots, \tag{2.16}$$

*where*

$$\omega^*(d_\vartheta(v)) = \begin{cases} \omega(d_\vartheta(v)), & \text{if } v \text{ is proper}, \\ \sqrt{\omega(d_\vartheta(v))}, & \text{if } v \text{ is virtual}. \end{cases}$$

Next we discuss the monotonicity of the spectrum of the generator with respect to the coupling constants. Notice that the relaxation time $\tau_2$ is defined by $\tau_2 := (n(1-\mu_2))^{-1}$, where $\mu_2$ is the second largest eigenvalue of $A$. In other words, it is the inverse of the spectrum gap of the (continuous-time) generator of the Glauber dynamics. In the language of our operator $B$ in (2.8),

$$\tau_2 = (1-\lambda_2)^{-1}, \tag{2.17}$$

where $\lambda_2$ is the second largest eigenvalue of $B$ whose largest eigenvalue is 1 no matter how the coupling constants are given. Yuval Peres has conjectured that the relaxation time $\tau_2$ is an increasing function of the couplings $J_{xy}$.[7] Nacu has shown that it is true for cycles in Ref. 7. It is easy to see that in the Hilbert space $L^2(\pi)$, the constant function, which is the eigenfunction for the eigenvalue 1 of $A$, is orthogonal to $\mathcal{H}_n$ which is spanned by $e_i$'s. Moreover, Nacu has shown that the eigenvector for

$\lambda_2$ of $B$ is a linear combination of $e_i$'s. Thus Peres's conjecture means that when restricted to $\mathcal{H}_n$ the largest eigenvalue of $B$ increases as couplings increase.

**Remark 2.3.** Since the eigenvalues of $B$ and $\widetilde{B}$ are the same, the following relation holds:

$$\lambda_{\max}^2 = \lim_{m\to\infty} \frac{(e_0, \widetilde{B}^{2m+2} e_0)}{(e_0, \widetilde{B}^{2m} e_0)}. \tag{2.18}$$

Thus, we may try with this relation and Theorem 2.2 to check the monotonicity of $\lambda_{\max}$ w.r.t. couplings, but we do not have any good idea to see this.

Next we discuss a different type of monotonicity in the spectral analysis of the generator. We assume that the coupling constants vary with a parameter $t \in T$, where $T$ is an open interval containing 0, such that

$$J_{ii+1}(0) = J_{ii+1} \quad \text{and} \quad J'_{ii+1}(t) \geq 0, \quad \text{for } t \in T \text{ and } i = 0, 1, \ldots, n-1.$$

One is interested to see the monotonicity of moments. Our observation is that if the nearby couplings do not vary too much differently from each other, then all the moments are increasing as the coupling parameters increase. Let $\mu_0(t)$ be the distribution of the spectrum of $B$ with couplings $J_{ii+1}$ being replaced by $J_{ii+1}(t)$ for $i = 0, \ldots, n-1$. We denote the $m$th moment of $\mu_0(t)$ by $M_m(t) := M_m(\mu_0(t))$. Consider the following condition of relatively mild varying.

$$\frac{s_{i-1}s_i}{1 + c_{i-1}c_i} \leq \frac{J'_{ii+1}(t)}{J'_{i-1i}(t)} \leq \left(\frac{s_{i-1}s_i}{1 + c_{i-1}c_i}\right)^{-1}, \quad i = 0, 1, \ldots, n-1. \tag{2.19}$$

**Theorem 2.4.** *Suppose condition* (2.19) *holds. Then*

$$\frac{d}{dt} M_m(t) \geq 0 \quad \textit{for all } m \geq 0.$$

**Proof.** By Theorem 2.2 it is enough to show that $d\omega_i/dt \geq 0$ for all $i = 0, \ldots, n-1$. By direct computation we have

$$\begin{aligned}
\frac{d\omega_i}{dt} &= \frac{d}{dt}(\alpha_i \beta_i) \\
&= \frac{d}{dt} \frac{s_{i-1}^2}{(c_{i-2} + c_{i-1})(c_{i-1} + c_i)} \\
&= \frac{2s_{i-1}}{(c_{i-2} + c_{i-1})^2 (c_{i-1} + c_i)^2} \\
&\quad \times [(c_{i-1} + c_i)\{(1 + c_{i-2}c_{i-1}) J'_{i-1i}(t) - s_{i-2}s_{i-1} J'_{i-2i-1}(t)\} \\
&\quad + (c_{i-2} + c_{i-1})\{(1 + c_{i-1}c_i) J'_{i-1i}(t) - s_{i-1}s_i J'_{ii+1}(t)\}].
\end{aligned} \tag{2.20}$$

Thus $d\omega_i/dt \geq 0$ if the quantities in the brackets $\{\,\}$ are non-negative, and the condition (2.19) suffices for it. □

### 2.3. *Constant couplings*

From now on we assume that the coupling constants are uniform over the edges, i.e. for all $i \in V$,

$$J_{ii+1} \equiv J \text{ (constant)}. \tag{2.21}$$

Then (2.6) reads as

$$Ae_i = \left(1 - \frac{1}{n}\right) e_i + \frac{\gamma}{n}(e_{i-1} + e_{i+1}), \tag{2.22}$$

with

$$\gamma = \frac{1}{2}\tanh 2J. \tag{2.23}$$

Let $\mathcal{A}$ be the operator algebra generated by $A$ and the identity operator $I$ on $\mathcal{H}_n$ ($\mathcal{A}$ being understood as acting on $\mathcal{H}_n$).

In this Hilbert space we introduce a subspace, which we denote by $\Gamma_n$, with orthonormal system defined by the following:

$$\Phi_0 := e_0,\ \ \Phi_1 := \frac{1}{\sqrt{2}}(e_1 + e_{n-1}),\ \ \Phi_2 := \frac{1}{\sqrt{2}}(e_2 + e_{n-2}), \ldots,$$
$$\Phi_{[\frac{n}{2}]-1} = \frac{1}{\sqrt{2}}(e_{[\frac{n}{2}]-1} + e_{n-([\frac{n}{2}]-1)}),\ \ \Phi_{[\frac{n}{2}]} = \begin{cases} e_{\frac{n}{2}}, & n \text{ even},\\ \dfrac{1}{\sqrt{2}}(e_{[\frac{n}{2}]} + e_{[\frac{n}{2}]+1}), & n \text{ odd}. \end{cases} \tag{2.24}$$

From (2.22), it is not hard to see that $A$ can be represented on the Hilbert space $\Gamma_n$. Now for $\varepsilon \in \{+, \circ, -\}$, we define new operators $A^\varepsilon$ as follows:

$$\begin{aligned} &A^+\Phi_i = \sqrt{\omega_{i+1}}\Phi_{i+1}, \quad i = 0, 1, \ldots, \left[\frac{n}{2}\right] - 1, \quad A^+\Phi_{[\frac{n}{2}]} = 0,\\ &A^-\Phi_0 = 0, \quad A^-\Phi_i = \sqrt{\omega_i}\Phi_{i-1}, \quad i = 1, 2, \ldots, \left[\frac{n}{2}\right],\\ &A^\circ\Phi_i = \alpha_{i+1}\Phi_i, \quad i = 0, 1, \ldots, \left[\frac{n}{2}\right], \end{aligned} \tag{2.25}$$

where the coefficients $\{\omega_i\}_{i=1}^{[\frac{n}{2}]}$ and $\{\alpha_i\}_{i=1}^{[\frac{n}{2}]+1}$ are defined by

$$\begin{aligned} &\omega_1 = 2\left(\frac{\gamma}{n}\right)^2,\ \ \omega_2 = \cdots = \omega_{[\frac{n}{2}]-1} = \left(\frac{\gamma}{n}\right)^2,\ \ \omega_{[\frac{n}{2}]} = \begin{cases} 2\left(\frac{\gamma}{n}\right)^2, & n \text{ even},\\ \left(\frac{\gamma}{n}\right)^2, & n \text{ odd}, \end{cases}\\ &\alpha_1 = \cdots = \alpha_{[\frac{n}{2}]} = 1 - \frac{1}{n},\ \ \alpha_{[\frac{n}{2}]+1} = \begin{cases} 1 - \dfrac{1}{n}, & n \text{ even},\\ 1 - \dfrac{1}{n} + \dfrac{\gamma}{n}, & n \text{ odd}. \end{cases} \end{aligned} \tag{2.26}$$

It is clear that the operator $A$ is decomposed as

$$A = A^+ + A^\circ + A^-. \tag{2.27}$$

Hereby we have obtained the so-called interacting Fock space $(\Gamma_n, \{\omega_i\}, A^+, A^\circ, A^-)$ with Jacobi parameters $(\{\omega_i\}, \{\alpha_i\})$.[3] As usual we call (2.27) as quantum decomposition of the operator $A$. The operators $A^+$, $A^0$ and $A^-$ are called the creation, conservation, and annihilation operators in that order.

Let us now briefly recall the orthogonal polynomials for the distribution on the real line. Let $\mu$ be a probability measure on the real line that has all moments. Let $\{P_0, P_1, \ldots\}$ be the Gram–Schmidt orthogonalization of the vectors $\{1, x, x^2, \ldots\} \subset L^2(\mathbb{R}, \mu)$. When $P_i$'s are normalized as

$$P_i(x) = x^i + \cdots, \quad i = 0, 1, \ldots,$$

it is well known that there is a pair of sequences $\{\omega_i\}_{i=1}^{\infty}$ and $\{\alpha_i\}_{i=1}^{\infty}$ such that (see Ref. 3)

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x - \alpha_1, \\ xP_i(x) &= P_{i+1}(x) + \alpha_{i+1}P_i(x) + \omega_i P_{i-1}(x), \quad i = 1, 2, \ldots. \end{aligned} \tag{2.28}$$

The pair of sequences $(\{\omega_i\}_{i=1}^{\infty}, \{\alpha_i\}_{i=1}^{\infty})$ are called the Jacobi coefficients of the orthogonal polynomials associated with $\mu$.

Our first result is the following.

**Theorem 2.5.** *Let* $(\{\omega_i\}_{i=1}^{[\frac{n}{2}]}, \{\alpha_i\}_{i=1}^{[\frac{n}{2}]+1})$ *be the coefficients defined in* (2.26). *There is a probability measure* $\mu_n$ *on* $\mathbb{R}$ (*which has only finite number of point masses*) *such that* $(\{\omega_i\}_{i=1}^{[\frac{n}{2}]}, \{\alpha_i\}_{i=1}^{[\frac{n}{2}]+1})$ *is the Jacobi coefficients of* $\mu_n$ *and satisfies for all* $m = 1, 2, \ldots$

$$M_m(\mu_n) := \int_{-\infty}^{\infty} x^m \mu_n(dx) = (\Phi_0, A^m \Phi_0) = (\Phi_0, (A^+ + A^\circ + A^-)^m \Phi_0).$$

**Proof.** The given pair of coefficients are a Jacobi coefficients of finite type.[3] By the action of the operators $A^+$, $A^\circ$, and $A^-$ in (2.25), the result follows from the general theory. See Theorems 1.52 and 1.54 of Ref. 3. □

## 3. Growing Cycles and Limit Distribution

In this section we discuss the growing cycles and the limit distribution of the generator of the Glauber dynamics. In order to have a nontrivial limit, we need some rescaling. Let us write the operator $A$ in (2.6) by $A^{(n)}$ to indicate that it is the generator of the dynamics for the system of size $n$. We introduce a new operator by

$$B^{(n)} := nA^{(n)} - (n-1)I. \tag{3.1}$$

So, $B^{(n)}$ is a dilation of $A^{(n)}$ followed by a translation. By looking Eq. (2.6) and following the method in Sec. 2.2, it is obvious that a quantum decomposition and the interacting Fock space for the operator $B^{(n)}$ correspond to the Jacobi coefficients $(\{\omega_i^{(n)}\}_{i=1}^{[\frac{n}{2}]}, \{\alpha_i^{(n)}\}_{i=1}^{[\frac{n}{2}]+1})$ defined by

$$\begin{aligned} &\omega_1^{(n)} = 2\gamma^2,\ \omega_2^{(n)} = \cdots = \omega_{[\frac{n}{2}]-1}^{(n)} = \gamma^2,\ \omega_{[\frac{n}{2}]}^{(n)} = \begin{cases} 2\gamma^2, & n \text{ even}, \\ \gamma^2, & n \text{ odd}, \end{cases} \\ &\alpha_1^{(n)} = \cdots = \alpha_{[\frac{n}{2}]+1}^{(n)} = 0. \end{aligned} \tag{3.2}$$

Clearly, as $n$ goes to infinity the Jacobi coefficients $(\{\omega_i^{(n)}\}_{i=1}^{[\frac{n}{2}]}, \{\alpha_i^{(n)}\}_{i=1}^{[\frac{n}{2}]+1})$ converge to the limit $(\{\omega_i\}_{i=1}^{\infty}, \{\alpha_i\}_{i=1}^{\infty})$ defined by

$$\begin{aligned} &\omega_1 = 2\gamma^2, \quad \omega_2 = \omega_3 = \cdots = \gamma^2 \\ &\alpha_1 = \alpha_2 = \cdots = 0. \end{aligned} \tag{3.3}$$

Notice that the distribution corresponding to the above Jacobi coefficients is the so-called Kesten distribution $\mu_{2\gamma^2,\gamma^2}$ with parameters $2\gamma^2$ and $\gamma^2$.[3] We summarize the above story into a theorem.

**Theorem 3.1.** *Let $B^{(n)}$ be defined by* (3.1). *Then the distribution of $B^{(n)}$ in the vacuum state converges to a Kesten distribution $\mu_{2\gamma^2,\gamma^2}$ as $n$ goes to infinity.*

**Remark 3.2.** The Kesten distribution $\mu_{2\gamma^2,\gamma^2}$ has a density[3]: $\mu_{2\gamma^2,\gamma^2}(dx) = \rho_\gamma(x)dx$. The function $\rho_\gamma(x)$ is given by

$$\rho_\gamma(x) = \begin{cases} \dfrac{1}{\pi}\dfrac{1}{\sqrt{4\gamma^2 - x^2}}, & |x| < 2\gamma, \\ 0, & |x| > 2\gamma. \end{cases}$$

For the Kesten distribution $\mu_{2,1}$, the $m$th moment $\int x^m \mu_{2,1}(dx)$ is the number of one-dimensional random walk paths of length $m$ that starts at the origin and returns to the origin. Since the adjacency matrix of the graph of infinite chain has this distribution,[3] we conclude that as the cycle increases, the generator of the Glauber dynamics, after being suitably dilated and translated, behaves much more the same as the adjacency matrix of infinite chain.

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